

## Entropy functions with 5D Chern-Simons terms

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## Entropy functions with 5D Chern-Simons terms

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**ABSTRACT:** In this article we reconsider Sen's entropy function analysis for 5D supergravity actions containing Chern-Simons terms. The apparent lack of gauge invariance is usually tackled via a 4D reduction. Here we motivate how a systematic 5D procedure also works. In doing so, it becomes important to identify the correct 5D charges. In particular, we perform explicit calculations for the black ring and 5D black hole. In the black ring analysis, we find Chern-Simons induced spectral flow shifts emerging out of Sen's formalism. We find that the entropy function nevertheless remains gauge invariant and the resulting electric charges are identified as Page charges. For the black hole too, 5D gauge invariance is confirmed. Our 5D analysis enables us to fix a mismatch that arose in the electric charges of Goldstein and Jena's 4D-reduced calculation. Finally we provide an interpretation for the  $e^0 \leftrightarrow p^0$  exchange in the entropy function as an interpolation between black hole and black ring geometries in Taub-NUT.

**KEYWORDS:** Black Holes in String Theory, Supergravity Models

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## 1 Introduction

The entropy function formalism of Sen [1, 2] allows for a very systematic approach to computing black hole entropy in  $D$  dimensions with  $AdS_2 \times S^{D-2}$  near-horizon geometry, especially including higher derivative corrections. Subsequently this formalism has also found application to other extremal black objects such as black rings and even black holes with reduced near-horizon isometry groups [11, 12]. However, in odd dimensions, the presence of Chern-Simons terms in the supergravity action no longer leaves the latter invariant under large gauge transformations; whereas Sen’s original construction was formulated for gauge as well as reparametrization invariant actions. To overcome this hurdle, it was proposed in [13] to perform a dimensional reduction in order to bring the Lagrangian density into a gauge invariant form and then apply the entropy function method. Therefore whilst computing the black ring entropy function, the authors of [11] first perform a dimensional reduction of the 5D supergravity Lagrangian into a gauge invariant 4D Lagrangian, upon which the standard entropy function method can then be applied.

In this work we revisit the black ring and 5D static black hole entropy functions. Instead of taking recourse to a dimensional reduction, we propose that a meaningful 5D computation of the entropy function with Chern-Simons terms is possible.<sup>1</sup> While performing such a 5D analysis, a key issue which requires careful consideration is how we should treat charges in 5D and their corresponding spectral flows. For the benefit of our esteemed reader, let us recall that these are also the same questions that have been at the center of much debate [9, 10, 27–29] with regards to the 4D/5D conjecture for black holes and black rings [14, 15]. It is not surprising that those subtleties also come into play when trying to perform an intrinsic 5D analysis of the entropy function formalism. And that happens

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<sup>1</sup> In this paper we only consider gauge-type Chern-Simons terms. Presumably our considerations are valid for gravitational or mixed gauge-gravitational Chern-Simons terms as well.

because the introduction of Chern-Simons terms brings in three different notions of charge: Brane-source charge, Maxwell charge and Page charge [33]. Which one is more relevant depends very much on the details of the geometric configuration one is interested in. Then expressing the entropy function in terms of the correct 5D charges will turn out to be the crucial step towards resolving its apparent lack of gauge invariance. We do this explicitly first for the black ring and then for the black hole.

In case of the black ring, even though we find that the reduced action is no longer invariant under large gauge transformations, it nevertheless turns out that the entropy function itself does remain gauge invariant. Furthermore we show that this invariance is no coincidence, but stems from an underlying spectral flow symmetry of the theory, which leaves the entropy function invariant under spectral flow transformations. In order to achieve this, we have to first demonstrate how the relevant spectral flow relations emerge within the 5D computation whilst solving the equations of motion in the presence of Chern-Simons terms. Through this we shall also be able to identify the 4D/5D dictionary, using which the 4D-reduced computation of Goldstein and Jena [11] can be recovered — except for one subtle issue on which our 5D computation differs from their 4D computation for reasons that will become clear in the calculations that follow.

In this context it is worth pointing out to the work of [17] on  $AdS_5$  black holes in gauged supergravity where it was also suggested that Chern-Simons terms would somehow facilitate charge shifts of the form  $q_I \rightarrow q_I + c_I$ . However these authors propose a modified Sen's formalism with shifted charges directly implemented and the  $c_I$  being undetermined shift parameters. Then in [18] this issue was pushed further (see also [19] for work in a related context), where they propose a new entropy function for rotating 5D black holes in order to extract asymptotic charges from near horizon data. However the above attempts do not work for black ring type geometries. The philosophy we adopt in this work is that it is not necessary to modify Sen's formalism by imposing charge redefinitions ad hoc, but rather a consistent 5D evaluation of Sen's functional is possible and from which these charge shifts can be seen to emerge in a natural way. We will see that this is indeed the case and such charge shifts carry a natural interpretation as spectral flow shifts in 5D. This way we are able to uniquely determine the shift parameters and unlike previous attempts our procedure works simultaneously for both  $AdS_2$  as well as  $AdS_3$  near horizon factors.

After having treated the black ring, we proceed to check gauge invariance of the 5D black hole entropy function. Here again we see that a 5D calculation shows some interesting differences when compared to the 4D calculation of [11]. This will have something to do with the  $x^\mu$ -dependence of the moduli  $a^I$  (which are  $\psi$ -components of the 5D gauge fields  $A^I$ ). In the calculation of [11], the  $x^\mu$ -dependence of  $a^I$  are retained throughout dimensional reduction of Chern-Simons terms to 4D and only then are they set to be constants. Apparently this is what seems to create a seemingly incorrect shift in electric charges when comparing their result for the black hole entropy to that of [20]. Here we claim that the way out is not to assume such a coordinate dependence (which would even be incompatible with the isometries of the 5D near-horizon geometry) in a 5D calculation. In addition to finding an agreement with the result of [20], our claim also leads to the correct 5D electric charges which are seen to perfectly tally with recent results of [35], who perform an explicit near-horizon analysis pertaining to 5D supergravity.

The outline of this paper is as follows. In section 2 we compute the black ring entropy function without dimensional reduction. The 5D charges turn out to be Page charges, which exhibit spectral flow behaviour. The entropy function however is shown to be spectral flow invariant. Section 3 concerns gauge invariance of the 5D black hole entropy function. For both black objects, we compare the 5D charges computed here via the 5D entropy formalism to those computed in the supergravity analysis of [35]. In section 4 we clarify the subtleties in charges arising between explicit 4D and 5D applications of the entropy function. Then in section 5 we provide an interpretation for the  $e^0 \leftrightarrow p^0$  switch within the entropy formalism as corresponding to a black hole  $\leftrightarrow$  black ring interpolation in supergravity. Finally in section 6 we conclude with some discussions.

## 2 Spectral flow invariance of the black ring entropy function

Let us now perform a 5D computation of the black ring entropy function and derive the associated spectral flow relations from the equations of motion therein.

Consider the action of 5D minimal ungauged two-derivative supergravity theory coupled to  $N - 1$  abelian vector multiplets. Writing only the bosonic fields, we have

$$\mathcal{S}_5 = \frac{1}{16\pi G_5} \int R * 1 - G_{IJ} dX^I \wedge *dX^J - \frac{1}{2} G_{IJ} F^I \wedge *F^J - C_{IJK} A^I \wedge F^J \wedge F^K \quad (2.1)$$

where  $X^I$  are massless scalars parameterizing the five dimensional “very special geometry”. These scalars define the compactification volume  $\mathcal{V}$  via the relation

$$C_{IJK} X^I X^J X^K = \mathcal{V} \quad (2.2)$$

The couplings  $G_{IJ}$  are functions of the scalar moduli and are defined as

$$G_{IJ} = -\frac{1}{2} \frac{\partial}{\partial X^I} \frac{\partial}{\partial X^J} \ln \mathcal{V} \Big|_{\mathcal{V}=1} \quad (2.3)$$

The indices  $I, J, K$  run from 1 to  $N$  while  $C_{IJK}$  is a completely symmetric tensor and  $F^I = dA^I$  are  $N$   $U(1)$  gauge fields.

Now let us consider the effect of large gauge transformations to the action in eq. (2.1). These transformations can be parametrised as

$$A^I \longrightarrow A^I + \Lambda^I \quad (2.4)$$

where  $\Lambda^I$  are one-forms whose components we shall shortly specify. Clearly the Chern-Simons term in the action is not invariant under large gauge transformations.<sup>2</sup> In fact large gauge transformations introduce integral shifts of the action that pick up a phase in the path integral. In this section, we revisit the black ring entropy function and show that

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<sup>2</sup>Small gauge transformations pose no problems in this case. This is because the extra gauge terms in the action can be expressed as an integral of a total derivative which is then evaluated as a surface term at infinity, where the gauge parameters asymptotically vanish. However, with large gauge transformations this is not so. The latter are not obtained as continuous transformations from the identity element and hence cannot be expressed as exact forms that could be partially integrated and evaluated as surface terms.

instead of the 4D approach followed by [11], one can also perform an alternate well-defined 5D calculation. Consequently, we need to directly tackle the problematic Chern-Simons terms above; which we do so by invoking spectral flow shifts.

To begin with, the 5D geometry is expressed via a Kaluza-Klein ansatz for an  $AdS_2 \times S^2 \times S^1$  topology (metric in eq. (2.8) below). Eventually of course, when one extremises the entropy function, the  $S^1$  fibres over the  $AdS_2$  (see [11]) precisely recovering the known near-horizon  $AdS_3 \times S^2$  metric ([3, 4, 6, 7]) of a supersymmetric black ring. Also the 5D gauge potential  $A^I$  is expressed in terms of the aforementioned Kaluza-Klein decomposition as follows

$$A^I = A^I_\mu dx^\mu + a^I (d\psi + A^0_\mu dx^\mu) \tag{2.5}$$

where  $\psi$  parametrises the  $S^1$  circle with a periodicity of  $4\pi$ ; the  $A^0_\mu$  are off-diagonal entries in the 5D Kaluza-Klein metric (which we shall write down shortly); the scalars  $a^I$ , which are  $\psi$ -components of the 5D gauge potential  $A^I$ , are interpreted as axions in 4D; while  $A^I_\mu$  would just be the usual gauge potential in the four non-compact dimensions. Typically a large gauge transformation applied to an on-shell gauge potential can be implemented by choosing  $\Lambda^I = k^I d\psi$  where  $k^I$  are integral constants. Note however that  $A^I$  in eq. (2.5) is not yet on-shell since we have still to insert the values of  $a^I$ ,  $A^I_\mu$  and  $A^0_\mu$  after solving their respective equations of motion. We therefore write down a more general ansatz for the gauge parameter given by  $\Lambda^I = k^I (d\psi + A^0_\mu dx^\mu)$ . This can be implemented in eq. (2.5) via a simple shift

$$a^I \longrightarrow a^I + k^I \tag{2.6}$$

where the  $k^I$  are again integral constants. A few comments are in order here. Though eq. (2.6) still represents a shift in the  $\psi$ -component of  $A^I$ , this quantity ( $a^I$ ) also enters as a factor in the other  $x^\mu$ -components making it natural to allow shifts of  $K^I A^0_\mu$  in those respective components. Also it turns out, as will be clear in what follows, that eq. (2.6) in fact denotes the most general shift that correctly generates the full 5D spectral flow of charges. Moreover this choice of  $\Lambda^I$  will also leave the components of the on-shell field strength  $F^I$  independent of  $k^I$  once we solve the equation of motion for  $a^I$  and insert it into  $dA^I$ . These will be consistency checks of eq. (2.6) that we shall verify along the way.

The reduced action (terminology not to be confused with dimensionally reduced action) is now defined by integrating the 5D lagrangian density over  $S^2 \times S^1$  - the spatial horizon of the black ring, spanned by  $\theta$ ,  $\phi$  and  $\psi$

$$\mathcal{F}_5^{br} = \frac{1}{16\pi G_5} \int_\Sigma d\theta d\phi d\psi \sqrt{-g_5} \mathcal{L}_5 \tag{2.7}$$

Our task then is to evaluate  $\mathcal{F}_5^{br}$  in the background of the Kaluza-Klein metric for an  $AdS_2 \times S^2 \times S^1$  near-horizon topology

$$ds^2 = \omega^{-1} \left[ v_1 \left( -r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] + \omega^2 (d\psi + A^0_\mu dx^\mu)^2 \tag{2.8}$$

with  $A_\mu^I$  and  $A_\mu^0$  specified by

$$A_\mu^I dx^\mu = e^I r dt + p^I \cos\theta d\phi \quad A_\mu^0 dx^\mu = e^0 r dt \quad (2.9)$$

Here we take  $\omega, v_1, v_2, X^I, a^I, e^I, e^0$  to be constants in the near-horizon geometry. These will eventually be fixed as functions of the black ring charges upon extremisation.  $\omega$  is the radius of the Kaluza-Klein circle;  $v_1, v_2$  denote the  $AdS_2$  and  $S^2$  radii respectively;  $p^I$  are magnetic charges and  $e^I$  denote the corresponding electric fields in 4D (we shall soon write down the electric fields in 5D as well).  $e^0$  is dual to the magnetic field associated to a  $p^0$  charge (or D6-brane charge). However for rings, it is well known that the  $p^0$  charge is absent in the immediate vicinity of the horizon. In 4D,  $e^0$  too is treated as an electric field; however in 5D it will turn out to be associated to the angular momentum of the black ring along the  $S^1$  direction.

Putting all this together, and computing the 5D reduced action gives

$$\begin{aligned} \mathcal{F}_5^{br}(v_1, v_2, \omega, X^I, a^I, e^I, e^0, p^I) = & \left( \frac{2\pi}{G_5} \right) \left[ v_1 - v_2 + \frac{v_2 \omega^3 (e^0)^2}{4v_1} \right. \\ & \left. - \frac{v_1}{v_2} \omega \frac{G_{IJ}}{2} p^I p^J + \frac{v_2}{v_1} \omega \frac{G_{IJ}}{2} (e^I + \tilde{a}^I e^0)(e^J + \tilde{a}^J e^0) \right] \\ & + \left( \frac{24\pi}{G_5} \right) C_{IJK} [(e^I + \tilde{a}^I e^0) p^J \tilde{a}^K] \end{aligned} \quad (2.10)$$

We get the three terms in the first line of eq. (2.10) by computing the five dimensional Ricci scalar; the second line comes from the 5D Yang-Mills term in the action; and the last line is obtained from the Chern-Simons term. It is important to note that this result here differs from that of [11] on two counts.<sup>3</sup> Firstly we have shifts in the moduli  $a^I \rightarrow \tilde{a}^I \equiv (a^I + k^I)$ , which essentially encode large gauge transformations in 5D and consequently leave  $\mathcal{F}_5^{br}$  with a gauge ambiguity, which is manifest through the explicit  $k^I$  dependence. In a 4D-reduced calculation these shifts do not appear. The second point on which  $\mathcal{F}_5^{br}$  differs from its dimensionally reduced version  $\mathcal{F}_4^{br}$  is a factor of  $\frac{1}{2}$  in one of the two Chern-Simons contributions to the reduced action. This can be seen in the last line of eq. (3.7) in ref. [11] (note that their  $p^0$  has to be set to zero when considering black rings). In a 5D calculation, the reduced action  $\mathcal{F}_5^{br}$  does not contain this factor. In section 4 we shall see that this difference of factors arises because of the way the moduli  $a^I$  have been treated in a 5D calculation as opposed to how they were dealt with in the 4D case. This point will also turn out to be crucial in determining the correct 5D charges and in the end we shall justify our results by comparing with the analysis in [35].

Now, the 5D entropy function is defined as the Legendre transform of  $\mathcal{F}_5^{br}$  with respect to electric charges  $Q_I^{br}, Q_0^{br}$

$$\mathcal{E}_5^{br} = 2\pi \left[ Q_0^{br} e^0 + Q_I^{br} e^I - \mathcal{F}_5^{br}(v_1, v_2, \omega, X^I, \tilde{a}^I, e^I, e^0) \right] \quad (2.11)$$

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<sup>3</sup>Our  $G_{IJ}$  equals  $2f_{IJ}$  in the notation of [11].

where  $Q_I^{br}$  and  $Q_0^{br}$  are canonically conjugate to  $e^I$  and  $e^0$  respectively<sup>4</sup>

$$Q_I^{br} = \frac{\partial \mathcal{F}_5^{br}}{\partial e^I} \quad Q_0^{br} = \frac{\partial \mathcal{F}_5^{br}}{\partial e^0} \quad (2.12)$$

As we shall soon see,  $Q_I^{br}$ ,  $Q_0^{br}$  are 5D Page charges and are physical observables of the black ring. These charges will differ from the 4D electric charges  $q_I$  respectively  $q_0$  computed in [11].

Obtaining the entropy of a black ring then entails extremisation of the entropy function  $\mathcal{E}_5^{br}$  with respect to its moduli variables

$$\frac{\partial \mathcal{E}_5^{br}}{\partial a^I} = \frac{\partial \mathcal{E}_5^{br}}{\partial v_1} = \frac{\partial \mathcal{E}_5^{br}}{\partial v_2} = \frac{\partial \mathcal{E}_5^{br}}{\partial \omega} = \frac{\partial \mathcal{E}_5^{br}}{\partial X^I} = 0 \quad (2.13)$$

But before that let us see how the gauge ambiguity in the reduced action  $\mathcal{F}_5^{br}$ , and consequently in the entropy function  $\mathcal{E}_5^{br}$ , can be resolved. For that purpose we will need to know exactly how the Chern-Simons terms in  $\mathcal{F}_5^{br}$  affect physical charges  $Q_I^{br}$  and  $Q_0^{br}$ . It turns out that they induce spectral flow shifts in these charges. And we want to know how these shifts can be manifestly derived within the framework of the entropy function formalism itself. Consequently we shall see how  $\mathcal{E}_5^{br}$  remains invariant under these shifts.

We begin evaluating eq. (2.12) for  $Q_I^{br}$  and  $Q_0^{br}$  by making use of  $\mathcal{F}_5^{br}$  from eq. (2.10). To avoid cluttering of notation let us normalise the  $\frac{4\pi}{G_5}$  factors in front of the charges to 1. Later in the final result we shall restore these constants. We then get the following expressions

$$Q_I^{br} = \left(\frac{v_2}{v_1}\right) \omega \frac{G_{IJ}}{2} (e^J + e^0 \tilde{a}^J) + 6C_{IJK} \tilde{a}^J p^K \quad (2.14)$$

and

$$Q_0^{br} = \left(\frac{v_2}{v_1}\right) \left(\frac{1}{4} \omega^3 e^0 + \omega \frac{G_{IJ}}{2} \tilde{a}^I (e^J + e^0 \tilde{a}^J)\right) + 6C_{IJK} \tilde{a}^I \tilde{a}^J p^K \quad (2.15)$$

That these are in fact the correct 5D charges for a black ring can be checked by comparing these expressions to the 5D Page charges recently computed in the supergravity analysis of [35], who showed that the near-horizon region of a black ring also encodes full information of its charges measured at asymptotic infinity. The results of [35] yield

$$Q_I^{\text{Page}} = \frac{1}{16\pi^2} \int_{\Sigma} *F_I + 6C_{IJK} A^J \wedge F^K \quad (2.16)$$

$$Q_0^{\text{Page}} = -\frac{1}{16\pi^2} \int_{\Sigma} *d\xi + *(\xi \cdot A^I) F_I + 6C_{IJK} (\xi \cdot A^I) A^J \wedge F^K \quad (2.17)$$

where  $\Sigma$  is a 3-cycle over the spatial horizon. For the black ring  $\Sigma$  specialises to  $S^2 \times S^1$ .  $\xi$  denotes the axial Killing vector with respect to the  $\psi$ -direction, while  $(\xi \cdot A^I)$  is an inner

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<sup>4</sup>Formally the  $Q_I^{br}$  can be expressed as conjugates to  $(e^I + \tilde{a}^I e^0)$ . However since the Jacobian between the four and five dimensional electric variables ( $e^I$  respectively  $e^I + \tilde{a}^I e^0$ ) is one, we end up with the first expression in eq. (2.12).



product between a vector field and a one-form. The Killing field  $\xi$  generates isometries along the  $\psi$ -direction; leading to a conserved charge, which is simply the angular momentum. In fact, the right-hand side of eq. (2.17) is just the Noether charge of Wald. Page charges are in fact not gauge invariant (due to an explicit  $A^I$ -dependence in these expressions), even though they are conserved and localised [33]. Now in order to strike a comparison between these charges of [35] and those computed here using the 5D entropy formalism, we will need to explicitly integrate the right-hand sides of eqs. (2.16) and (2.17). Since these are simply local integrations, it is sufficient to make use of only near-horizon data of the gauge fields and metric from eqs. (2.5) and (2.8). Computing the non-vanishing components of the 5D field strength gives  $F_{rt}^I = e^I + \tilde{a}^I e^0$  and  $F_{\theta\phi}^I = -p^I \sin\theta$ . In the near-horizon terminology, the axial vector  $\xi^i$  is found to be  $A_0^i$ , with non-vanishing components  $A_0^t = \frac{\omega^3 e^0}{v_{1r}}$  and  $A_0^\psi = -1$ . Using this we can determine  $F_0$ , which is just  $d\xi$ ; and by  $d\xi$  we mean  $\partial_i \xi_j dx^i \wedge dx^j$ . Note also that in the  $\xi \cdot A^I$  term, it makes sense to only consider the projection of the Killing field on the physical (on-shell sector) gauge fields. Putting together all these quantities and inserting them into eqs. (2.16) and (2.17) exactly reproduces eqs. (2.14) and (2.15). Hence we see that  $Q_I^{br}$  and  $Q_0^{br}$  obtained from the entropy function indeed represent the correct five-dimensional supergravity Page charges  $Q_I^{\text{Page}}$  and  $Q_0^{\text{Page}}$  respectively.

Now in the entropy function formalism the 5D field  $A^I$  in eq. (2.5) depends on three different moduli  $e^I$ ,  $e^0$  and  $a^I$ . Extremising  $\mathcal{E}_5^{br}$  with respect to these moduli and plugging the extremum values of these moduli back into eq. (2.5) basically determines the near-horizon gauge fields of the black ring.  $A^I$  can then be expressed purely in terms of electric and magnetic charges. For our purposes, these three extremisation conditions will fully determine the physical charges that source these gauge fields  $A^I$ . Hence eqs. (2.14) and (2.15) require further input from

$$\frac{\partial \mathcal{E}_5^{br}}{\partial a^I} = 0 \quad \implies \quad F_{rt}^I = 0 \tag{2.18}$$

and this exactly corresponds to  $\int_{\Sigma} *F_I = 0$  computed in [35] by explicit near-horizon integration. Physically, eq. (2.18) signifies a vanishing electric flux in the near-horizon geometry, which is simply what one would expect in the absence of a compact 3-cycle when the topology is that of  $AdS_3 \times S^2$ . Moreover the above result also tells us that the only non-vanishing on-shell components of the field strength (in this case the  $F_{\theta\phi}^I$ ) are all indeed gauge invariant.

We are now ready to identify the black ring spectral flow shifts that emerge from within the structure of the entropy function formalism itself. Separating the  $k^I$  dependence in  $Q_I^{br}$  and  $Q_0^{br}$  yields

$$Q_I^{br} = q_I + 6C_{IJK} k^J p^K \tag{2.19}$$

and

$$Q_0^{br} = q_0 + 2k^I q_I + 6C_{IJK} k^I k^J p^K \tag{2.20}$$

where  $q_I$  and  $q_0$  are read-off from eqs. (2.14) respectively (2.15) after replacing  $\tilde{a}^I$  by  $a^I$ ; and they can indeed be identified as the four dimensional (gauge invariant as well) electric

charges that appeared in the calculation of [11]. In 5D however,  $Q_I^{br}$  and  $Q_0^{br}$  are the correct physical observables [10, 35, 39].

Let us now determine what the conserved quantities, under spectral flow shifts of  $Q_I^{br}$  and  $Q_0^{br}$  look like. It is easy to see that  $\widehat{Q}_0$  defined by

$$\widehat{Q}_0 \equiv Q_0^{br} - C^{IJ} Q_I^{br} Q_J^{br} \tag{2.21}$$

is left invariant under spectral flow transformations described in eqs. (2.19) and (2.20) in the following sense

$$\widehat{Q}_0(Q_0^{br}, Q_I^{br}) = \widehat{Q}_0(q_0, q_I) \tag{2.22}$$

where  $C^{IJ} \equiv [C_{IJ}]^{-1}$  and  $C_{IJ} \equiv 6C_{IJK} p^K$ . Consequently the quantity  $\widehat{Q}_0$  is completely independent of the shift parameters  $k^I$  and this fact will play an important role in maintaining invariance of the 5D black ring entropy function.

Putting together all the above ingredients into eq. (2.11) gives us the entropy function in terms of 5D variables

$$\mathcal{E}_5^{br} = \frac{4\pi^2}{G_5} \left\{ v_2 - v_1 + \frac{v_1}{v_2} \left[ \omega \frac{G_{IJ}}{2} p^I p^J + 4\omega^{-3} (\widehat{Q}_0)^2 \right] \right\} \tag{2.23}$$

The first term in the square brackets in  $\mathcal{E}_5^{br}$  comes from the magnetic flux, while the second term is related to the effective momentum of D0-particles.<sup>5</sup> This brings us to the main result of this section that  $\mathcal{E}_5^{br}$  is indeed invariant under spectral flow transformations, once the moduli of the gauge field  $A^I$  have been determined. Here we have obtained  $\mathcal{E}_5^{br}$  in eq. (2.23) from a 5D calculation, and this agrees with the structural form of the dimensionally reduced  $\mathcal{E}_4^{br}$  of [11] because of spectral flow invariance.<sup>6</sup> Note however that while the form of the expression in eq. (2.23) is the same as that obtained in the 4D calculation of [11], their  $\widehat{Q}_0$  differs from ours in eq. (2.21) obtained above by a half in the last term. In section 4 we shall see that this is because of a slight discrepancy that enters the charges defined in [11]. Nevertheless the final 4D and 5D entropies reconcile despite the fact that  $\mathcal{F}_5^{br}$  and  $\mathcal{F}_4^{br}$  differ due to explicit gauge transformation parameters and also that the observable 5D charges are Page charges whereas the 4D ones are Maxwell [35, 39]. This illustrates the point that for a 5D action which includes Chern-Simons terms, there is another way besides a dimensional reduction to 4D; a direct 5D calculation will also give the correct result once the right 5D variables have been implemented into the calculation. Note that  $\mathcal{E}_5^{br}$  is not yet an entropy and here what we see is that even when  $\mathcal{E}_5^{br}$  is not at its stationary point, it is still gauge invariant. Hence we get

$$\mathcal{E}_5^{br}(Q_0^{br}, Q_I^{br}, p^I, v_1, v_2, \omega, X^I) = \mathcal{E}_5^{br}(q_0, q_I, p^I, v_1, v_2, \omega, X^I) \tag{2.24}$$

upon inserting eqs. (2.19) and (2.20) into eq. (2.23). The left-hand side is what one gets from an explicit 5D calculation, whereas the right-hand side is what results from a dimensionally reduced computation.

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<sup>5</sup> These are precisely the left-movers of the dual (0, 4) SCFT [25].

<sup>6</sup> The 4D/5D lift for black rings is in fact a special case of spectral flow transformations when the value of  $k^I$  is set to  $p^I$  [39].

A 5D calculation is necessary to illustrate the inherent spectral flow associated to a black ring geometry. The physical interpretation of spectral flow for black rings has been discussed in [39]. The 4D/5D transformations themselves are in fact a special case of spectral flow transformations. And that is actually the reason why application of the entropy function formalism to black rings should work well either in 4D or 5D (even though we think that an explicit 5D computation expresses charge/geometric data more naturally).

For the sake of completeness, let us also extremise with respect to the remaining moduli, as in eq. (2.13); and show that the resulting black ring entropy obtained from our 5D calculation indeed gives the right answer. Solving for  $v_1, v_2, \omega$  gives

$$v_1 = v_2 = \omega \frac{G_{IJ}}{2} p^I p^J + 4\omega^{-3} (\widehat{Q}_0)^2 \tag{2.25}$$

and

$$\omega^4 = \frac{12(\widehat{Q}_0)^2}{\frac{G_{IJ}}{2} p^I p^J} \tag{2.26}$$

and upon using these values of  $v_1, v_2, \omega$  back into  $\mathcal{E}_5^{br}$  yields<sup>7</sup>

$$\mathcal{E}_5^{br} = \frac{8\pi^2}{G_5} \sqrt{\left(\frac{2G_{IJ}}{3} p^I p^J\right)^{\frac{3}{2}} \widehat{Q}_0} \tag{2.27}$$

Of course the couplings  $G_{IJ}$ , which are functions of the yet-to-be-extremised scalar moduli  $X^I$ , will depend on geometric data of the specific compactification space. For our purposes we leave it with the general expression in eq. (2.27).

### 3 Gauge invariance of the 5D black hole entropy function

We now repeat our calculation for the 5D black hole. The near-horizon metric ansatz is again taken to be  $AdS_2 \times S^2 \times S^1$ . However this time round it turns out that the  $S^1$  fibres over the  $S^2$ , eventually leading to an  $AdS_2 \times S^3$  geometry near the horizon [11]. It has been proven in [36–38] that even in the rotating case, the near-horizon isometry of an extremal black hole contains an  $SO(2, 1)$  symmetry. Moreover, that the entropy function formalism can also be applied to such rotating black holes having  $AdS_2$  isometry was shown in [34]. Such a black hole in 5D carries a Kaluza-Klein monopole charge  $p^0$ , which comes from uplifting a D6-brane in Type II A theory to M-theory and the black hole sits at the origin of the KK monopole.<sup>8</sup> Even though this geometric configuration is different from that of a black ring, it is still reasonable to implement the Kaluza-Klein metric ansatz of eq. (2.8)

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<sup>7</sup> In this result the charge of the Kaluza-Klein monopole  $p^0$  is taken to be unity. This corresponds to a black ring in Taub-NUT (or flat space whenever the Taub-NUT radius goes to infinity). The case  $p^0 > 1$  corresponds to taking an orbifold of the Taub-NUT and that in turn leads to a near-horizon factor of  $AdS_3/\mathbb{Z}_{p^0}$  for the black ring. In the notation of [11] this charge has been denoted as  $\tilde{p}^0$  and like in that work its effect can be included by modding the  $S^1$  circle by  $\mathbb{Z}_{p^0}$ .

<sup>8</sup> Note that when  $p^0 > 1$ , the  $S^1$  circle is modded by  $\mathbb{Z}_{p^0}$  consequently giving an  $AdS_2 \times S^3/\mathbb{Z}_{p^0}$  near-horizon geometry. This shall be appropriately implemented in what follows.

provided the off-diagonal components  $A_\mu^0$  are suitably modified for the black hole case. We consider the same type of black hole as in [11], so that the results of our analysis can be compared to theirs. Hence  $A_\mu^0$  is taken as

$$A_\mu^0 dx^\mu = p^0 \cos\theta d\phi \tag{3.1}$$

where  $p^0$  denotes the Kaluza-Klein monopole charge. Note also that the quantity  $e^0$  is absent for these black holes, which corresponds to an absence of Kaluza-Klein momentum  $J_0^{KK}$ . Here  $J_0^{KK} = 0$  is only to be thought of as vanishing of the intrinsic angular momentum (resulting from the absence of D0-charge in the brane bound state). In [11] it was claimed that this black hole is static. However there is a slight subtlety to that.<sup>9</sup> The effective angular momentum is in fact non-vanishing. As a quick check one can easily compute the integral in eq. (2.17) and we see that the second term in the integrand carries a non-vanishing contribution. Nevertheless it will turn out that this effective contribution does not enter the entropy formula (and this last point was presumably the reason that this black hole was viewed as a static system in [11]). On the other hand a black hole of the BMPV type [23], is a true rotating black hole with an angular momentum that enters the entropy formula. Such a black hole would be obtained had we started with a bound state of spinning M2's in Taub-NUT (or a D0-D2-D6 bound state in Type II A). Instead what we have here is a black hole more of the type discussed in [24]. It can be conceived as a bound state of non-rotating M2's sitting at the tip of a Taub-NUT-flux geometry (D2-D4-D6 in II A), where the intrinsic angular velocity of the horizon vanishes, leaving only the flux induced component of the angular momentum which affects the geometry but not the entropy formula - in some sense like a static black hole in a flux background.

Within this set-up we now compute  $\mathcal{F}_5^{bh}$  to get

$$\begin{aligned} \mathcal{F}_5^{bh}(v_1, v_2, \omega, X^I, a^I, e^I, p^I, p^0) = & \left( \frac{2\pi}{G_5} \right) \left[ v_1 - v_2 - \frac{v_1 \omega^3 (p^0)^2}{4v_2} \right. \\ & + \frac{v_2}{v_1} \omega \frac{G_{IJ}}{2} e^I e^J - \frac{v_1}{v_2} \omega \frac{G_{IJ}}{2} (p^I + \tilde{a}^I p^0)(p^J + \tilde{a}^J p^0) \left. \right] \\ & + \left( \frac{24\pi}{G_5} \right) C_{IJK} [(p^I + \tilde{a}^I p^0) e^J \tilde{a}^K] \end{aligned} \tag{3.2}$$

which differs from eq. (2.10) with the replacement  $p^I \rightarrow p^I + \tilde{a}^I p^0$  and a  $(p^0)^2$  term in the 5D Ricci scalar that replaces the  $(e^0)^2$  term in the black ring computation. Just as in the black ring analysis before, we once again find that  $\mathcal{F}_5^{bh}$  computed here is not exactly going to be the same as  $\mathcal{F}_4^{bh}$  in [11]. Firstly, in a 5D approach the gauge parameters  $k^I$  show up and secondly, the relative factors in front of the Chern-Simons contributions will differ from those in the 4D computation of [11] (refer to eq. (3.7) in ref. [11] after setting  $e^0 = 0$  therein). Once again in  $\mathcal{F}_5^{bh}$  this factor does not appear. In the next section we shall see in detail how this affects the definition of electric charges in 5D and thereby fix a small mismatch, with respect to the definition of 5D charges, in the result for the entropy obtained by [11] when compared to that of [20].

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<sup>9</sup>We are grateful to Dumitru Astefanesei for a discussion on this point.

Having eq. (3.2) in hand, we are now in a position to write the 5D black hole charges from the analog of the definition in eq. (2.12)

$$Q_I^{bh} = \left(\frac{v_2}{v_1}\right) \omega \frac{G_{IJ}}{2} e^J + 6C_{IJK} (p^J + \tilde{a}^J p^0) \tilde{a}^K \quad (3.3)$$

Moreover using

$$\frac{\partial \mathcal{E}_5^{bh}}{\partial a^I} = 0 \quad \implies \quad p^I + \tilde{a}^I p^0 = 0 \quad (3.4)$$

we can write eq. (3.3) as

$$Q_I^{bh} = \frac{1}{16\pi^2} \int_{\Sigma} *F_I \quad (3.5)$$

since  $F_{rt}^I = e^I$  and  $F_{\theta\phi}^I = -(p^I + \tilde{a}^I p^0) \sin\theta$ . Here  $\Sigma$  is now an  $S^3$ , the spatial horizon of the black hole. Eq. (3.4) is just the condition for vanishing of the effective magnetic flux

$$\int_{S^2} F_I = 0 \quad (3.6)$$

in other words suggesting the absence of a compact 2-cycle in this black hole geometry. Eq. (3.4) also confirms that all the non-vanishing on-shell components of the field strength are gauge invariant. Moreover for given magnetic charges  $p^I$  and  $p^0$ , the constraint  $p^I + \tilde{a}^I p^0 = 0$  imposes a restriction on the value of  $k^I$ . Therefore for this black hole, we cannot set-up arbitrary spectral flow shifts for the charges.

In the terminology of [33], eq. (3.5) implies that  $Q_I^{bh}$  is not a Page but a Maxwell charge,<sup>10</sup> which is gauge invariant and does not show spectral flow behaviour.  $Q_I^{bh}$  therefore represents the same physical observable in 5D as well as in 4D alike.

Under these considerations, the entropy function for this black hole takes the form

$$\mathcal{E}_5^{bh} = \frac{4\pi^2}{G_5} \left\{ v_2 - v_1 + \frac{v_1}{v_2} \left[ \frac{1}{4} \omega^3 (p^0)^2 + \omega^{-1} 2G^{IJ} Q_I^{bh} Q_J^{bh} \right] \right\} \quad (3.7)$$

where  $G^{IJ}$  is defined as the inverse of  $G_{IJ}$ . Once again we have obtained a gauge invariant entropy function from an explicit 5D calculation in terms of physical 5D variables. Now it is straightforward to extremise  $\mathcal{E}_5^{bh}$  with respect to  $v_1$ ,  $v_2$  and  $\omega$  to get

$$v_1 = v_2 = \frac{1}{4} \omega^3 (p^0)^2 + \omega^{-1} 2G^{IJ} Q_I^{bh} Q_J^{bh} \quad (3.8)$$

and

$$\omega^4 = \frac{8G^{IJ} Q_I^{bh} Q_J^{bh}}{3(p^0)^2} \quad (3.9)$$

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<sup>10</sup> Additionally, in this case the Maxwell charge is localised within  $\Sigma$  and does not require integration over all space because the source term  $F^J \wedge F^K$  in the 5D supergravity equation of motion:  $d * F_I = -6C_{IJK} F^J \wedge F^K$ , vanishes following eq. (3.4).

Then eliminating  $v_1$ ,  $v_2$  and  $\omega$  by way of substituting their values at the stationary point back into  $\mathcal{E}_5^{bh}$  leaves us with

$$\mathcal{E}_5^{bh} = \frac{4\pi^2}{G_5} \sqrt{p^0 \left( \frac{8G^{IJ}}{3} Q_I^{bh} Q_J^{bh} \right)^{\frac{3}{2}}} \tag{3.10}$$

which finally gives us the entropy of this black hole. The couplings  $G^{IJ}$  can be determined depending on the specific choice of compactification. Here  $Q_I^{bh}$  is the observable electric charge in 5D and since we have shown above that this charge does not exhibit any spectral flow behaviour, it exactly equals the number of M2-branes wrapping Calabi-Yau 2-cycles. Upon shrinking the M-theory circle and reducing to Type II A, the M2-branes directly descend to D2-branes. Then  $Q_I^{bh}$  is also the physical charge for a 4D black hole.

#### 4 Comparing charges between 4D/5D approaches

In this section we demonstrate how the charge mismatch, obtained in [11] when compared to that of [20], is fixed by our 5D approach. We then provide the necessary consistency checks. Firstly comparing eq. (3.10) above to the entropy obtained by [20] (whose computation is performed via a 5D attractor mechanism), indeed gives an exact agreement; thereby fixing the mismatch in the result of [11] where the charges  $Q_I^{bh}$  in the entropy formula were shifted by  $3C_{IJK}p^J p^K / p^0$  (refer eq. (3.41) in ref. [11] where in their notation  $\hat{q}_I$  enters the entropy formula rather than  $q_I$ ; then in eq. (3.65) in the same reference they compare  $\hat{q}_I$  to the charge in [20] where the latter itself does not contain any shift terms). In our case, using eqs. (3.3) and (3.4) we see that the charges entering the entropy are  $Q_I^{bh} = \left(\frac{v_2}{v_1}\right) \omega \frac{G_{IJ}}{2} e^J$  without any  $p^I$  dependence. The extra  $3C_{IJK}p^J p^K / p^0$  terms in [11] do not enter our  $Q_I^{bh}$  and consequently the match to [20] is exact.

Before we delve into reasons underlying this mismatch, let us at this stage perform a consistency check for our charges computed above. We want to see whether  $Q_I^{bh}$  compares to the charge integral obtained in the supergravity analysis of [35], which would serve as an independent verification. For that purpose consider eq. (2.16) with  $\Sigma$  taken to be an  $S^3$ . Since we know the near-horizon components of  $A^I$  and  $F^I$ , we insert these into eq. (2.16) and evaluate the integral. Because  $F_{\theta\phi}^I = 0$ , the  $\int_{\Sigma} A^J \wedge F^K$  part of the integral vanishes and the  $\int_{\Sigma} *F_I$  term precisely reproduces  $\left(\frac{v_2}{v_1}\right) \omega \frac{G_{IJ}}{2} e^J$ . That verifies that our expression for  $Q_I^{bh}$  in eqs. (3.3) and (3.4) is indeed the correct electric charge of the black hole.

One may now ask why the charges of [11] picked up those incorrect shifts? Which may be rephrased by asking what went wrong with their Chern-Simons contributions to the 4D reduced action  $\mathcal{F}_4^{bh}$ ? The Chern-Simons terms in  $\mathcal{F}_4^{bh}$  were obtained from a four dimensional reduction of the 5D supergravity action (refer appendix A in [11]). This then gave rise to the abovementioned factor of  $\frac{1}{2}$  in  $\mathcal{F}_4^{bh}$  (eq. (3.7) in [11]), which subsequently lead to an erroneous shift in their definition of charges. In our calculation the factor of  $\frac{1}{2}$  did not appear in the 5D reduced action  $\mathcal{F}_5^{bh}$  and that gave the correct electric charge, which matches [20] and confers with [35]. This subtle difference in a factor of  $\frac{1}{2}$  between the reduced actions computed in [11] and that computed here seems to be related to how we treated the moduli  $a^I$  in our calculation, as opposed to how the same was handled in [11].

There they assume an  $x^\mu$ -dependence for the moduli  $a^I$ , while performing a dimensional reduction of Chern-Simons terms. These  $a^I$  are set to constants only when one arrives at the four dimensional set-up. Subsequently the four dimensionally reduced Chern-Simons Lagrangian density (see (A.11) in ref. [11]) picks up a factor of half in front of the second term therein. This is how the  $\frac{1}{2}$  enters the 4D reduced action  $\mathcal{F}_4^{bh}$  and consequently the charges. On the other hand, in our 5D calculation, in the absence of any dimensional reduction there is no natural way to assume an  $x^\mu$ -dependence for  $a^I$  (whilst already in the 5D near-horizon geometry) and then suddenly set them to constants at some other stage of the calculation. The 5D components of the field strength  $F_{rt}, F_{\theta\phi}$  are constants in the near-horizon geometry and giving the fields an  $x^\mu$ -dependence through  $a^I$  would tantamount to a deformation of the near-horizon geometry and possibly interfere with the AdS isometries which were crucial to the formulation. Therefore in our calculations we have set all 5D near-horizon moduli as constants (whose values are determined upon extremisation) throughout the calculation and this procedure seems to give the correct answers. It would be interesting to see if this four dimensional reduction can also be redone keeping all the  $a^I$  constant and then check if that leads to the right charges. The focus of this paper however was to show that an explicit five dimensional entropy function calculation also works and provides us with consistent answers.

Finally let us remark that the much-discussed factor of  $\frac{1}{2}$  in the 4D reduced actions affects the charge definitions of both the black ring as well as the black hole. However in case of the black hole the effect is far more drastic. Let us clarify this point. We start with the ring. Suppose that the factor of half had also appeared in our reduced action as a coefficient of the last term in eq. (2.10) (multiplying  $C_{IJK}e^0\tilde{a}^I\tilde{a}^Jp^K$ ). That would only change the charge  $Q_0^{br}$  in eq. (2.15) by replacing the '6' in the last term ( $C_{IJK}e^0\tilde{a}^I\tilde{a}^Jp^K$ ) with a '3'. The term itself does not vanish when comparing to  $Q_0^{\text{Page}}$  in eq. (2.17). Of course this changes the numerical coefficients in eqs. (2.20) and (2.21). But eq. (2.22) will still be satisfied for the modified equations and a new  $\widehat{Q}_0$  (with a half instead of a one in the last term in eq. (2.21)) will finally enter the black ring entropy function in eq. (2.23). Hence the changes in this case only show up as different coefficients of existing terms. But in the case of the black hole the erroneous factor in the reduced action adds another term to the charge which we clearly know does not exist. We can see this as follows. Suppose the last term in eq. (3.2) (the one with a  $p^0$ ) carried a half. This would carry forward as an extra numerical factor in the definition of  $Q_I^{bh}$  in eq. (3.3). However after using eq. (3.4) (the factor half does not affect this equation because the  $C_{IJK}$  terms do not enter  $\mathcal{E}_5^{bh}$ ) in eq. (3.3), we are left with an extra  $3C_{IJK}p^Jp^K/p^0$  term in the definition of  $Q_I^{bh}$ . And as mentioned above, this extra term neither confers to the Page charge integrals in [35] nor to the literature in [20]. Hence the changes are far more conspicuous in case of the black hole.

## 5 Switching over $e^0 \leftrightarrow p^0$ as a form of black hole — black ring interpolation

Earlier in section 2 we saw how the near-horizon solution of a black ring can be expressed via various moduli parameters. Among these  $e^I$  and  $e^0$  are conjugate to the electric charges and

angular momentum respectively, while the magnetic flux  $p^I$  is a fixed quantity. On the other hand, the 5D black hole of section 3 only carried electric variables  $e^I$  and fixed magnetic variables  $p^I, p^0$ . From the perspective of the entropy function formalism, obtaining the metric of a black hole from that of a black ring can simply be achieved by switching off the  $e^0$  contribution to the metric and turning on a  $p^0$  one instead (and then extremising with respect to these new moduli). This assignment was first proposed in [11], where it appears as an ad hoc choice that reproduces the leading order entropies of the two black objects. In this section we want to provide a physical justification for this assignment of parameters. We will soon see that switching the terms  $e^0 r dt \leftrightarrow p^0 \cos\theta d\phi$  among each other in the near-horizon Kaluza-Klein metric will in fact be equivalent to changing the modulus  $l$  (here  $l$  is the three dimensional distance of the black ring from the origin of the Taub-NUT base space) from a specified finite quantity to a vanishing limit in the complete 5D supergravity solution. Gravitationally this means we are shrinking the 5D black ring to the origin of the base space to get a 5D black hole. In this sense, we argue that the  $e^0 \leftrightarrow p^0$  switch is actually a black hole — black ring interpolation rather than some sort of black hole — black ring duality, that was suggestively speculated in [11]. Let us now examine this in more detail.

In section 2 we demonstrated that  $p^I, Q_I$  and  $Q_0$  computed from a 5D entropy function analysis, are the correct physical observables of a black ring. Moreover a glance at the microscopic description of a black ring as a bound state of branes will in fact reveal that the observable charges are not exactly the brane charges [8, 9]. Microscopically a black ring can be described by a Calabi-Yau compactification of M-theory on a circle [26] with M2-M5 branes wrapping 2- respectively 4-cycles on the Calabi-Yau. The remaining one leg of the M5-brane wraps the M-theory circle thus giving a black string along this  $S^1$  (as in the description of [25]). This string is stabilised by angular momentum modes running along the circle. The relation between brane charges and observable charges in fact takes the form [10]

$$\begin{aligned}
 q_I^{M2} &= Q_I - 6C_{IJK} p_{M5}^J p_{M5}^K \\
 p_{M5}^I &= p^I \\
 J_0^{KK} &= Q_0 - p_{M5}^I q_I^{M2} - 6C_{IJK} p_{M5}^I p_{M5}^J p_{M5}^K
 \end{aligned}
 \tag{5.1}$$

These shifts from the actual brane charges have been shown in [39] to be manifestations of spectral flow when  $k^I = p^I$ . In this way the above relations also serve as a 4D/5D map between the two-center system of a D0-D2-D4 black hole in 4D, placed in the vicinity of a D6-charge; and a black ring in 5D. Hence when the M-theory circle shrinks to zero size then the charge shifts due to spectral flow disappear and the brane charges  $J_0^{KK}, q_I^{M2}, p_{M5}^I$  (which now become D0, D2, D4 charges respectively in the Type II A description) coincide with the observable charges. Having stated the relations between physical and brane charges of the black ring, we can now incorporate these into supergravity solutions.

In order to study a supergravity construction that interpolates between 5D black holes and black rings in its different limits, we start by considering the most general 5D  $\mathcal{N} = 1$  ungauged supergravity solution [21, 22] which is given by the following 5D metric and



gauge fields

$$\begin{aligned}
 ds_5^2 &= -f^2(dt + \Omega)^2 + f^{-1}ds^2(M_4) \\
 F^I &= d[fX^I(dt + \Omega)] - \frac{2}{3}fX^I(d\Omega + \star d\Omega)
 \end{aligned}
 \tag{5.2}$$

where  $X^I$  are scalar fields in abelian vector multiplets. They satisfy the constraint equation  $C_{IJK}X^IX^JX^K = 1$  and  $X_I$  are defined by the condition  $X^IX_I = 1$ .  $ds^2(M_4)$  above refers to the Gibbons-Hawking metric of a 4D hyper-Kahler base space, which in our case is simply taken to be  $ds^2(TN)$ , the Taub-NUT metric (or  $ds^2(\mathbb{R}^4)$  when considering a black ring in flat space) having KK-monopole charge. Let  $r, \theta, \phi, \psi$  denote coordinates on the 4D base space with  $(r, \theta, \phi)$  locally parameterising an  $\mathbb{R}^3$  and  $\psi$  running along a compact  $S^1$  with periodicity  $4\pi$ . The Hodge dual  $\star$  is taken with respect to the 4D base space. The function  $f$  and the one-form  $\Omega$  can then be determined in terms of four harmonic functions  $H_{TN}(x), K^I(x), L_I(x)$  and  $M(x)$  (with  $x \in \mathbb{R}^3$ ) in the following sense

$$\begin{aligned}
 f^{-1}X_I &= \frac{1}{4}H_{TN}^{-1}C_{IJK}K^JK^K + L_I \\
 \Omega &= \left( -\frac{1}{8}H_{TN}^{-2}C_{IJK}K^IK^JK^K - H_{TN}^{-1}L_IK^I + M \right) \\
 &\quad \times (d\psi + \cos\theta d\phi) + \widehat{\Omega}
 \end{aligned}
 \tag{5.3}$$

where  $\widehat{\Omega}$  is defined by

$$\nabla \times \widehat{\Omega} = H_{TN}\nabla M - M\nabla H_{TN} + K^I\nabla L_I - L_I\nabla K^I
 \tag{5.4}$$

Operating the gradient on both sides of this equation yields integrability conditions

$$H_{TN}\nabla^2 M - M\nabla^2 H_{TN} + K^I\nabla^2 L_I - L_I\nabla^2 K^I = 0
 \tag{5.5}$$

which are evaluated at each pole (charge center) in  $\mathbb{R}^3$ .

Within the above framework, a supergravity solution for any black object is now reduced to the task of specifying four harmonic functions. Let us first write these down for a black ring and then we shall see how to interpolate them to a black hole solution. For a black ring we have the following

$$\begin{aligned}
 H_{TN}(x) &= \frac{4}{R_{TN}^2} + \frac{p_{KK}^0}{|x|} & L_I(x) &= v_I + \frac{q_I^{M2}}{|x-l|} \\
 K^I(x) &= \frac{p_{M5}^I}{|x-l|} & M(x) &= v_0 + \frac{J_0^{KK}}{|x-l|}
 \end{aligned}
 \tag{5.6}$$

Here  $p_{KK}^0$  is the charge of the Kaluza-Klein monopole in M-theory, which reduce to  $p_{KK}^0$  D6-branes in Type II A. The case  $p_{KK}^0 = 1$  corresponds to a Taub-NUT, otherwise the 4D hyper-Kahler base space is an orbifold of Taub-NUT, such that its geometry in the neighbourhood of the origin is of the type  $\mathbb{C}^2/\mathbb{Z}_{p_{KK}^0}$ . Let us clarify the remaining notation as well:  $R_{TN}$  denotes the asymptotic radius of the original Taub-NUT;  $x \in \mathbb{R}^3$  and  $l$  is a modulus in  $\mathbb{R}^3$  which denotes the distance between the plane containing the  $S^1$  of the

ring and the origin of base space.  $v_I$  is a constant determined at infinity and  $v_0$  will soon get fixed via the integrability conditions. These harmonic functions have been specified via brane charges in the system. The bound states of branes wrapping Calabi-Yau cycles form BPS point particles in  $\mathbb{R}^3$  and the poles in the above harmonic functions are attained precisely at the location of these BPS particles. The M2-M5- $J^{KK}$  particle sits at  $x = l$ , while the KK monopole is located at  $x = 0$ . From a 4D point of view this is a 2-center black hole system, but in 5D it's just a black ring in a Taub-NUT orbifold [15].

Now let us evaluate eq. (5.5) for the above harmonics at each of the two poles. This yields the following two integrability conditions

$$v_0 = -\frac{J_0^{KK}}{|l|} \tag{5.7}$$

$$J_0^{KK} = v_I p_{M5}^I \left( \frac{p_{KK}^0}{|l|} + \frac{4}{R_{TN}^2} \right)^{-1} \tag{5.8}$$

Physically this implies that  $J_0^{KK}$ ; which contributes part of the angular momentum along the  $\psi$ -direction of the ring; cannot be arbitrarily chosen, but is fixed for a given configuration. The above conditions can then be inserted back into eq. (5.6) and thereafter implementing the charge transformations in eq. (5.1) (which were obtained as spectral flow shifts from the supergravity action), essentially lays down the complete black ring solution. This compares to the standard solutions of [3, 4, 6, 7, 10] when expressed in more convenient coordinates — but we will not require that here.

Now let us study the behaviour of this black ring in the limit  $l \rightarrow 0$ . From [5] we already know that we should recover a 5D black hole in this limit. However the purpose of our presentation is to make a clear distinction between branes that constitute a black ring bound state from those that constitute a black hole bound state when the modulus  $l$  is driven to zero. Then we want to relate these brane charges to the spectral flow of those respective black objects in order to determine the physical charges.

Let us begin with eqs. (5.7) and (5.8). When  $l \rightarrow 0$ , they reduce to

$$J_0^{KK} = 0 \tag{5.9}$$

$$v_0 = -\frac{v_I p_{M5}^I}{p_{KK}^0} \tag{5.10}$$

and the harmonics in eq. (5.6) become

$$\begin{aligned} H_{TN}(x) &= \frac{4}{R_{TN}^2} + \frac{p_{KK}^0}{|x|} & L_I(x) &= v_I + \frac{q_I^{M2}}{|x|} \\ K^I(x) &= \frac{p_{M5}^I}{|x|} & M(x) &= -v_I p_{M5}^I \end{aligned} \tag{5.11}$$

after having used eqs. (5.9) and (5.10) therein. What we have now is a BPS configuration in which there is not only a KK monopole at the origin of the Taub-NUT orbifold, but also the M5-M2 charge is now bound to this monopole. Moreover these bound states of branes have vanishing  $J_0^{KK}$  charge. This is a 5D black hole (or a D2-D4-D6 black hole

from the point of view of a 4D reduction). Furthermore from the analysis in section 3 we saw that in the case of the 5D black hole, there are no spectral flow shifts. Therefore for this configuration, the brane charges  $p_{KK}^0$ ,  $p_{M5}^I$  and  $q_I^{M2}$  respectively correspond to the following physical charges

$$\begin{aligned} p_{KK}^0 &= p^0 \\ p_{M5}^I &= p^I \\ q_I^{M2} &= Q_I \end{aligned} \tag{5.12}$$

Now recalling the entropy function formalism, these charges are precisely associated to the following near-horizon variables:  $p^0$ ,  $p^I$ ,  $e^I$ . To sum up the contents of this section, we find that the physical interpretation of switching  $e^0$  with  $p^0$  in the entropy formalism's near-horizon ansatz corresponds to interpolating between limits of the modulus  $l$  on a Taub-NUT orbifold, which in supergravity yields an interpolation between black hole/black ring geometries. Moreover building this association to supergravity also serves the purpose of providing a justification for the specific choice of moduli in the Kaluza-Klein metric ansatz of [11], for each of the two geometries.

## 6 Discussion and conclusions

The inclusion of Chern-Simons terms in the entropy function formalism has rather been a bit of a puzzle due to its apparent lack of gauge invariance under large gauge transformations. This being because Sen's original derivation [1] was based on the premise of gauge and reparametrisation invariant lagrangian densities. The dimensional reduction approach was proposed [13] in order to rectify this. In view of the proposed 4D/5D connection [14, 15], that such a recipe works might not come as a total surprise though. However even in those developments several contentious subtleties stood out as regards the correct physical notion of charge in 4D and 5D [9, 10, 27–29]. In this note we have argued that there is no fundamental obstruction to a well-defined 5D treatment of entropy functions with Chern-Simons terms, provided one implements the correct physical 5D charges into the calculations. In general these 5D charges differ from those used in the dimensionally reduced approach due to spectral flow shifts. However to fully specify a charge, one needs to obtain the equation of motion of the corresponding gauge field which is sourced by that charge. Within the setting of the entropy formalism, these gauge fields are determined via moduli  $e^I$ ,  $e^0$  and  $a^I$ . Therefore upon extremising  $\mathcal{F}_5$  with respect to these moduli one can determine the electric charges. On the other hand the magnetic charges are pre-fixed from the beginning. Our calculations demonstrate that once the physical 5D charges are made manifest in the entropy function, it immediately falls into a 5D gauge invariant expression, even without requiring to fix all the remaining moduli  $v_1$ ,  $v_2$ ,  $\omega$ ,  $X^I$ . In other words we do not need to modify Sen's formalism, but only correctly identify the physical 5D charges and perform computations manifestly in terms of these charges. Moreover because of the fact that gauge fields and consequently charges of 5D geometries with different near-horizon topologies will in general be quite different, we find that one cannot construct a universal entropy function that describes

any 5D geometry in the presence of Chern-Simons terms and which is also gauge invariant. In reference [11], they do manage to write down a unified entropy function, however that can only be expressed in terms of off-shell charges and it is in fact not invariant under spectral flow transformations. Therefore in order to check 5D gauge invariance, we had to treat the  $AdS_2 \times S^2 \times S^1$  black ring topology and the  $AdS_2 \times S^3$  black hole topology separately.

As is well-known, Chern-Simons terms in odd dimensions induce spectral flow shifts in the supergravity action, which also reflect in the defining notion of charges in these theories [33]. In our analysis for the black ring, we have seen that these spectral flow equations also arise in a natural way out of Sen's formalism in 5D. Consequently the 5D electric charges were no longer gauge invariant and neither was the reduced action  $\mathcal{F}_5^{br}$ . Nonetheless the entropy function  $\mathcal{E}_5^{br}$  itself turned out to remain invariant under gauge/spectral flow transformations if it is expressed as a function of the correct physical charges. We have also verified that the electric charges computed here from Sen's approach are identical to the Page charges expected from 5D supergravity: our charges calculations for the black ring give a precise match with the charge integrals recently computed by [35] on the basis of near-horizon data.

On the other hand, whilst computing for the 5D black hole we found that the electric charges turned out not to be Page but simply 5D Maxwell charges with no spectral flow shifts. This was because a vanishing magnetic flux in an  $AdS_2 \times S^3$  geometry suppresses all spectral flow shifts. As a consequence, the 5D charges of this black hole exactly match those of its 4D counterpart upon compactification of the fifth dimension. This corroborates with the 4D/5D lift of [14]. Within this set-up, gauge invariance of the entropy function thereon follows in a straightforward manner. Then extremising  $\mathcal{E}_5^{bh}$  to compute the black hole entropy indeed gave us an exact match with the result of [20], where the latter was obtained via an attractor mechanism calculation. This resolves the slight discrepancy in the result of [11] where their entropy did not quite match [20]: because their electric charges did not agree with those of [20]. Besides the comparison to [20], we have also provided additional evidence to support the claim that  $Q_I^{bh}$  computed here are the correct charges to work with by showing that they also match exactly with the charges of [35], which were obtained from a 5D supergravity approach. The discrepancy in the charges of [11] arise whilst dimensionally reducing the Chern-Simons terms to 4D: namely, they assume an  $x^\mu$ -dependence for the moduli  $a^I$ ; and only set the  $a^I$  to constants in the final step. Consequently this introduces terms in  $\mathcal{L}_{CS}^{4D}$ , which incorrectly shift their electric charges, thereby causing a mismatch with the entropy of [20]. However from the point of view of a manifestly 5D calculation, there was no natural way to assume such an  $x^\mu$ -dependence (whilst already in the 5D near-horizon geometry) and then abruptly deem them constants later in the calculation. The 5D components of the field strength  $F_{rt}, F_{\theta\phi}$  are constants in the near-horizon geometry and giving the fields an  $x^\mu$ -dependence through  $a^I$  would seem to come in conflict with the isometries of the near-horizon geometry. Moreover from the result of [35] given in eq. (2.16), the  $\int_\Sigma 6C_{IJK}A^J \wedge F^K$  term vanishes for this black hole in the absence of an effective magnetic flux ( $p^I + \tilde{a}^I p^0$ ). It is only the  $\int_\Sigma *F_I$  term that contributes to the charge. Inserting the expression for the near-horizon field strength into the integral of eq. (2.16), exactly reproduces our expression for  $Q_I^{bh}$ . The extra terms in

the charges of [11] would simply not agree with the integral of [35]. This seems to suggest that assuming an  $x^\mu$ -dependence on any of the moduli in the near-horizon geometry and then setting them to constants after dimensional reduction might be suspect. Within the entropy formalism, the isometries of the geometry are crucial to the analysis and all physical quantities ought to obey these. This imposes restrictions on the moduli, which works well when the latter are deemed constants in this geometry at any stage of the analysis.

A related line of interest which we have investigated in this note concerns black ring  $\leftrightarrow$  black hole interpolation in the context of Sen's formalism. The idea behind such an interpolation between geometries has been familiar since the work of [5], where it was shown using black ring solutions from [3, 4]. For what we had in mind here, it was more convenient to reformulate this interpolation using the most general 5D  $\mathcal{N} = 1$  ungauged supergravity solution of [21, 22] and varying the Taub-NUT modulus  $l$  from a specified point to a vanishing limit. This way the structure of harmonic functions and brane wrappings associated to the two geometries is more readily manifest. The supergravity solution of course captures the global structure of the geometry, whereas the entropy formalism is only a near-horizon analysis. Therefore in principle it is not possible to construct a full-fledged interpolation of solutions using the latter. However we have still managed to show within the Sen formalism that upon interchanging off-diagonal entries in the Kaluza-Klein metric bearing  $e^0$  terms with those bearing  $p^0$  ones, yields algebraic data that can be compared to the limiting supergravity solutions in such a way that parameters in the Kaluza-Klein metric can be specifically associated to brane wrappings in the supergravity solution for both the black ring and black hole. In retrospect, this also lends some physical intuition to the ad hoc assignment of variables made in the black hole/black ring metric ansatz proposed in [11]. Our original motivation in studying this  $e^0 \leftrightarrow p^0$  exchange was in the hope of finding some sort of black ring/black hole duality loosely speculated by [11]. However within the context of our analysis, the  $e^0 \leftrightarrow p^0$  exchange seems to relate more with the idea of a geometric interpolation rather than any string or gravitational duality. There is though an interesting work by [40] which might be more in the direction of seeking such a string duality between 5D black holes and black rings. In that work, the authors propose a duality between microstate degeneracies of a D0-D2-D4 system with those of a D0-D2-D6 system on the same Calabi-Yau via a Fourier-Mukai transform. From a 5D perspective, this would lift to a black hole/black string duality. From our discussion in section 4, we have seen that the M-theory lift of a D2-D4-D6 system gives a 5D black hole, whereas a D0-D2-D4 system in the vicinity of a D6 charge, lifts to a black ring. It would therefore be quite interesting to see if a microscopic duality along the lines of [40] can also be constructed for this black hole/black ring system.

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